Ch. 1 Introduction

§1.1 Basic models and Direction Fields

What is a differential equation?

An equation of derivatives

How do they arise?

1. Observe something happen
2. Model process using mathematics

Ex Population dynamics : Turkeys

$p(t)$ population at time t

- birth rate
- death rate

$$\frac{dp}{dt} = 4p - 10$$

Direction Fields

Consider an arbitrary DE

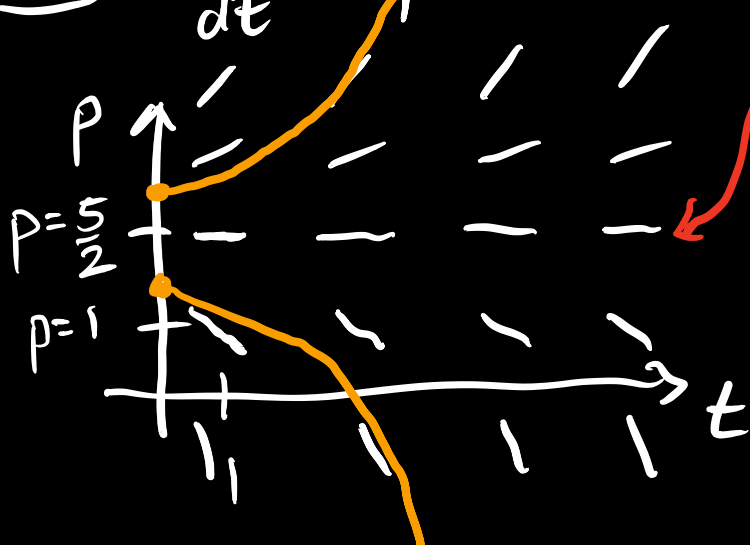
$$\frac{dy}{dt} = f(t,y)$$

- Evaluate f, which tells us $\frac{dy}{dt}$ at a point

  i.e. the slope of y

$\frac{dp}{dt} = 0$ equilibria

Ex $\frac{dp}{dt} = 4p - 10$



$(1,1)$

$$\frac{dp}{dt} = 4 - 10 = -6$$

$$\frac{dp}{dt} = 0 = 4p - 10$$

$$4p = 10$$

$$p = \frac{5}{2}$$

Problems with this model?

- gives negative population
- population goes to $\pm\infty$

→ go back to drawing board and cook-up an improved model

Building a Model

1. Identify independent and dependent variables
2. Choose convenient units
3. Describe what you are modeling and basic principles governing the problem

4. Translate 3 to variables from 1.

§1.2 Solutions of some DEs

<u>Ex</u> Object falling with air resistance

$$\underbrace{m\frac{dv}{dt}}_{\text{acceleration}} = \underbrace{mg}_{\text{gravitational force}} - \underbrace{\gamma v}_{\text{"air resistance"}}$$

"$ma = F$"

$$\frac{dv}{dt} = g - \frac{\gamma}{m}v$$

$$\frac{dv}{dt} = \frac{-\gamma}{m}\left(v - \frac{gm}{\gamma}\right) \quad (*)$$

$$\int \frac{dv}{dt}\,dt = \int \frac{-\gamma}{m}\left(v - \frac{gm}{\gamma}\right)dt$$

$$v = \frac{-\gamma}{m}\int v\,dt - \int g\,dt$$

$$\frac{\frac{dv}{dt}}{v-\frac{gm}{\gamma}} = \frac{-\gamma}{m}$$

$$\frac{d}{dt} f(g(x)) = f'(g(x))g'(x)$$

$$\int \frac{d}{dt} \ln\left(v-\frac{gm}{\gamma}\right) dt = \int \frac{-\gamma}{m}\, dt$$

$$\ln\left(v-\frac{gm}{\gamma}\right) = \frac{-\gamma}{m} t + C$$

arbitrary constant

→ get rid of C using the initial condition $v(0)=0$

→ $\ln\left(v-\frac{gm}{\gamma}\right) = \frac{-\gamma}{m} t + C$ is called the general solution

- geometric representation is a family of curves called integral curves

$$\begin{cases} m\dfrac{dv}{dt} = mg - \gamma v \\ v(0) = 0 \end{cases}$$

is called an initial value problem.

§1.3 Classification of Differential Equations

Ordinary and Partial Differential equations

(d) $\qquad$ (∂)

Ordinary differential equations (ODE)

- derivatives are with respect to only 1 variable

<u>Ex</u> $\quad \dfrac{d^2Q}{dt^2} + \dfrac{dQ}{dt} + Q = 0$

or

$$Q'' + Q' + Q = 0$$

Notation: $Q^{(n)} = \frac{d^n Q}{dt^n}$

$$Q^{(1)} = \frac{dQ}{dt} = Q'$$

Partial differential equation (PDE)
- differentiating with respect to more than 1 variable

Ex Heat Equation

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

Systems of Differential Equations
- multiple unknowns → multiple DEs

<u>Ex</u> Lotka-Volterra (predator-prey)

Prey $\frac{dx}{dt} = ax - \underline{\alpha xy}$

Predator $\frac{dy}{dt} = -cy + \underline{\gamma xy}$

interaction terms

Order of a DE

- The order of a DE is the order of the highest derivative

<u>Ex</u> $\underbrace{y'''}_{\hookrightarrow \text{highest order derivative}} + yy' = t^4$

$\Rightarrow$ 3rd order ODE

General n^{th} order ODE is of the

form $$F(t, y, y', \ldots, y^{(n)}) = 0$$

and we always assume we may write $$y^{(n)} = f(t, y, y', \ldots, y^{(n-1)})$$

Linear and Nonlinear Equations

The ODE
$$F(t, y, y', \ldots, y^{(n)}) = 0$$
is called linear if F is linear. General nth order linear ODE is
$$a_0(t) y^{(n)} + a_1(t) y^{(n-1)} + \cdots + a_n(t) y = g(t)$$
$\rightarrow$ if the equation is not linear

called nonlinear

→ much harder to solve

→ can sometimes be approximated by linear equations (linearization)

<u>Ex</u> small angle approximation

$$\sin\theta \cong \theta \qquad \sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots$$

Solutions of DEs

Consider the interval $\alpha < t < \beta$ and n^{th} order ODE

$$y^{(n)} = f(t, y, y', \ldots, y^{(n-1)})$$

$\phi(t)$ is a solution if

$$\phi^{(n)} = f(t, \phi, \phi', \ldots, \phi^{(n-1)})$$

for every $t \in (\alpha, \beta)$.

<u>Ex</u> $ty' - y = \boxed{t^2}$, $y = 3t + t^2$

$$y' = 3 + 2t$$

$$ty' - y = t(3+2t) - (3t + t^2)$$
$$= \cancel{3t} + 2t^2 \cancel{- 3t} - t^2$$
$$= t^2 \checkmark$$

Finding solutions is often not easy, but one of the most useful tools is making an ansatz (i.e. educated guess). If your ansatz satisfies the DE, you've found a solution!

<u>Ex</u> $y'' + y = 0$

We want a function that gives negative of the original after two derivatives

$\rightarrow \sin(t), \cos(t)$

<u>Some questions</u>

1. Given a DE, when does a solution <u>exist</u>?

2. Given a DE, how many solutions does it have? Is <u>the</u> solution unique?

3. Given a DE, can we actually determine a solution?